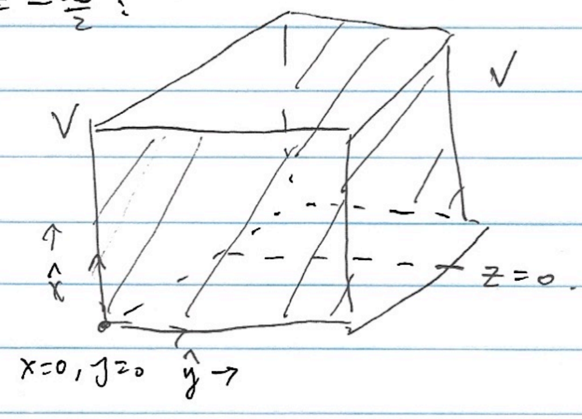
Jackson

2.23 (a)

Shift the cube on z-axis so that the boundaries resides at $z = +\frac{a}{2}$, $z = -\frac{a}{2}$:



This construction makes the problem about $z=0$ axis, then we have expansion:

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\sin(\beta_m y)\cosh(\gamma_{nm} z)$$

where $\alpha_n = \frac{n\pi}{a}$, $\beta_m = \frac{m\pi}{a}$, $\gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{a^2}}$, same formalism as developed in Jackson section 2.9, except now sinh is replaced with cosh because of different boundary conditions.

$\Phi|_{z=\frac{a}{2}} = V$, ~~so~~ this is to be utilized with orthogonality of sin to determine A_{nm}.

By orthogonality,

$$\int_0^a \int_0^a \Phi(x,y,z)\, dx\, dy \sin(\alpha_n x) \sin(\beta_m y) = A_{nm} \cosh(\gamma_{nm} z) \frac{4}{a^2}$$

$$\Rightarrow \quad A_{nm} = \frac{4}{a^2 \cosh(\gamma_{nm} z)} \int_0^a \int_0^a dx\, dy\, \Phi(x,y,z) \sin(\alpha_n x) \sin(\beta_m y)$$

At $z = \frac{a}{2}$, $\Phi(x,y) = V$, so this can be computed explicitly.

$$\int_0^a \int_0^a dx\, dy\, \Phi(x,y)\Big|_{z=\frac{a}{2}} \sin(\alpha_n x) \sin(\beta_m y)$$

$$= V \int_0^a \int_0^a \sin(\alpha_n x) \sin(\beta_m y)$$

$$= \begin{cases} V \left(\frac{2a}{n\pi}\right)\left(\frac{2a}{m\pi}\right) & n, m \text{ both odd} \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \quad A_{nm} = \frac{4}{a^2} \frac{1}{\cosh(\gamma_{nm} \frac{a}{2})} \frac{V}{1} \frac{2a}{n\pi} \frac{2a}{m\pi} \quad \text{if } n, m \text{ odd.}$$

$$= \boxed{\frac{16 V}{\pi^2 n m \cosh(\gamma_{nm} \frac{a}{2})}} \quad \leftarrow \; n, m \text{ both odd.}$$

Davidson Cheng
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